

QUANTUM COMPUTATION WITH SCATTERING MATRICES

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ABSTRACT. We discuss possible applications of the 1-D direct and inverse scattering problem to design of universal quantum gates for quantum computation. The potentials generating some universal gates are described.

1. INTRODUCTION

In this article we propose a theory of quantum scattering and notion of unitary scattering matrix to formulate quantum input-output relations. This differs from standard approach to this subject in which the quantum gates for quantum computing are considered to be unitary time evolution operator for a given, fixed time and with the Hamiltonian which describes the dynamics.

Representation of quantum gates as scattering matrices (S-matrices) may have a physical realization. In present paper we study one-dimensional scattering problem in electro-magnetic field. Varying electro-magnetic field and momentum of the electron we get a 1-parametric family of S-matrices. We prove that adjusting the electro-magnetic field we can deliver relevant 2-order and 4-order unitary matrix. It is well known that such matrices can be used as universal gates for quantum computing [8].

Another type of a scattering problem emerges in two-level quantum systems controlled by electric pulse. By a time-dependent Hamiltonian we also achieve the representation of desired gates as scattering matrices. In both approaches we use well-known inverse problem solutions to construct quantum gates.

Application of the quantum scattering process to quantum computing was studied in the works [5], [13], [14]. Articles [10], [11] are devoted to the monodromic approach to the quantum computing. In these works we construct the set of universal gates by monodromy matrices of Fuchsian connections on holomorphic vector bundles, such approach corresponds to holonomies of connections in the holonomic quantum

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computing [12], but holonomies do not depend on the path of integration. Unlike the monodromy matrices the scattering matrices are encountered in physical experiments, moreover there exists a one-to-one correspondence between the monodromy and scattering matrices [1] in one-dimensional case. In this article we describe potentials and corresponding S-matrices which generate the set of universal gates.

The paper is organized as follows. In section 2 we discuss possible applications of the 1-D electron scattering in an electro-magnetic potential to the problem of design of universal quantum gates for quantum computation. Using methodology of the inverse scattering problem we construct genuine potentials to obtain some kind of universal gates as a scattering matrix. In section 3 we consider the electric field interacting with two-level (or four-level) quantum systems. Transitions from initial state to final state are used as gates. The well known inverse scattering problem procedure allows to obtain potentials that maintain some universal gates. Finally we consider some potentials which allow to compute S-matrix as monodromy of the corresponding Fuchsian system.

2. SCATTERING ON THE LINE AND UNIVERSAL GATES

Let us consider the stationary Schrödinger equation on the line

$$(1) \quad \frac{d^2}{dx^2}\psi(x) + (k^2 + Q(x))\psi(x) = 0, \quad x \in (-\infty, \infty)$$

where $Q(x)$ is a continuous potential function vanishing at infinity, i. e. $Q(x) \rightarrow 0$, as $|x| \rightarrow \infty$.

Let $\varphi(x)$ be a solution of (1) which coincides with e^{-ikx} for $x \rightarrow -\infty$. Its complex conjugate function $\bar{\varphi}(x)$ also satisfies the equation (1) which coincides with e^{ikx} as $x \rightarrow -\infty$. Moreover we denote by $\psi(x)$ and $\bar{\psi}(x)$ the solution of (1) which coincides with e^{ikx} and e^{-ikx} respectively as $x \rightarrow \infty$. It is clear that there exists 2×2 matrix $M(k)$ such that

$$(2) \quad (\varphi(x), \bar{\varphi}(x)) = (\bar{\psi}(x), \psi(x))M(k).$$

It is known that $M(k) \in SU(1,1)$ and this matrix is called monodromy matrix [1]. The matrix $M(k)$ can be represented as

$$M(k) = \begin{pmatrix} a(k) & \bar{b}(k) \\ b(k) & \bar{a}(k) \end{pmatrix}, \quad |a(k)|^2 - |b(k)|^2 = 1.$$

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as an element of $SU(1, 1)$. Therefore we have

$$\begin{cases} \varphi = a\bar{\psi} + b\psi \\ \bar{\varphi} = \bar{b}\bar{\psi} + \bar{a}\psi \end{cases} \Rightarrow \begin{cases} \bar{\psi} = \frac{1}{a}\varphi - \frac{b}{a}\psi \\ \bar{\varphi} = \bar{b}(\frac{1}{a}\varphi - \frac{b}{a}\psi) + \bar{a}\psi = \frac{\bar{b}}{a}\varphi + \frac{1}{a}\psi \end{cases}.$$

Thus $(\bar{\psi}, \bar{\varphi}) = (\varphi, \psi)\tilde{S}(k)$, where $\tilde{S}(k) = \begin{pmatrix} \frac{1}{a} & \frac{\bar{b}}{a} \\ -\frac{\bar{b}}{a} & \frac{1}{a} \end{pmatrix}$. Similarly $(\bar{\varphi}, \bar{\psi}) = (\varphi, \psi)S(k)$, where $S(k) = \begin{pmatrix} \frac{\bar{b}}{a} & \frac{1}{a} \\ \frac{1}{a} & -\frac{\bar{b}}{a} \end{pmatrix}$. The matrices $\tilde{S}(k), S(k)$ are called [2], [3] the scattering matrices in various context.

We will use the notations $T(k) = \frac{1}{a(k)}$, $R(k) = \frac{b(k)}{a(k)}$ having an interpretation of transmission and reflection amplitudes respectively. The particle comes by left-to-right with impulse $k > 0$ goes through the barrier with transmission probability $|T(k)|^2$ and goes back with reflection probability $|R(k)|^2$, then output will be $e^{ikx} + be^{-ikx}$ on left from the barrier, and ae^{ikx} on the right from the barrier.

We consider here only the case $S(k)$. Such matrix is symmetric and unitary simultaneously (but it does not belong to $SU(2)$ in general) and many interesting gates can be represented in such form. Let us consider the mapping

$$\tau : SU(1, 1) \rightarrow U(2), \quad \begin{pmatrix} a & \bar{b} \\ b & \bar{a} \end{pmatrix} \mapsto \begin{pmatrix} \frac{\bar{b}}{a} & \frac{1}{a} \\ \frac{1}{a} & -\frac{\bar{b}}{a} \end{pmatrix}.$$

The image of this mapping $\tau(SU(1, 1))$ is the subset of the matrix in $U(2)$, which can be represented as scattering matrix.

Remark 1. Let initial states be denoted by $|0\rangle = e^{ikx}$, $|1\rangle = e^{-ikx}$.

If the scattering matrix has the form $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ then we obtain the states: $(|0\rangle + |1\rangle)\sqrt{2}$ and $(|0\rangle - |1\rangle)\sqrt{2}$.

Suppose that to write in quantum registers a natural number from 1 to x an n -qubit is needed. We take n -th tensor product of Hadamard gates and obtain:

$$\begin{aligned} (H \otimes \dots \otimes H)|0\rangle \otimes \dots \otimes |0\rangle &= \frac{1}{\sqrt{2^n}} H|0\rangle \otimes H|0\rangle \otimes \dots \otimes H|0\rangle = \\ &= \frac{1}{\sqrt{2^n}} (|0\rangle + |1\rangle) \otimes \dots \otimes (|0\rangle + |1\rangle) = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x\rangle. \end{aligned}$$

If $f : B^n \rightarrow B^n$ is a Boolean function, then it is well known that there exists a unitary operator U_f which computes all values of f in

the following way:

$$U_f\left(\frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x, 0\rangle\right) = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} U_f(|x, 0\rangle) = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n-1} |x, f(x)\rangle.$$

On the other hand by well known results (see [8]) there exist unitary matrices $U_2 \in U(2)$ and $U_4 \in U(4)$ such that each unitary matrix from $U(2^n)$ can be represented as a product of matrices $\mathbb{I} \otimes \dots \otimes U_2 \otimes \dots \otimes \mathbb{I}$ and $\mathbb{I} \otimes \dots \otimes U_4 \otimes \dots \otimes \mathbb{I}$. Such U_2, U_4 are called universal gates. Our aim is to represent the universal gates as S-matrices.

Example 1. The so called Hadamard matrix $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ can be represented as $\tau \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$. Similarly the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ has a representation $\tau \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Example 2. The complex numbers $a = \sqrt{n^2 + 1}, b = n$ define

$$\tau \begin{pmatrix} \sqrt{n^2 + 1} & n \\ n & \sqrt{n^2 + 1} \end{pmatrix} = \frac{1}{\sqrt{n^2 + 1}} \begin{pmatrix} n & 1 \\ 1 & -n \end{pmatrix}.$$

As $n \rightarrow \infty$ this matrix approximates the universal gate $NOT = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Similarly we can approximate more general gate $\begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}$, if we take $a = \sqrt{n^2 + 1}, b = -ne^{i\phi/2}$. Indeed, we have

$$T = \frac{1}{a} = \frac{1}{\sqrt{n^2 + 1}} e^{i\phi/2}, \quad \frac{b}{a} = \frac{n}{\sqrt{n^2 + 1}} e^{i\phi}, \quad -\frac{\bar{b}}{a} = \frac{n}{\sqrt{n^2 + 1}}$$

and

$$\begin{pmatrix} \frac{1}{\sqrt{n^2 + 1}} & \frac{1}{\sqrt{n^2 + 1}} e^{i\phi/2} \\ \frac{1}{\sqrt{n^2 + 1}} e^{i\phi/2} & \frac{n}{\sqrt{n^2 + 1}} e^{i\phi} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & e^{i\phi} \end{pmatrix}, \quad \text{as } n \rightarrow \infty.$$

□

It is known that the Hadamard matrix H and controlled phase gate $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix}$ can be used as a universal set of gates, as well as H

and controlled NOT $cNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Remark 2. There exists a mapping $SU(1, 1) \rightarrow SU(2)$

$$\begin{pmatrix} a & \bar{b} \\ \bar{b} & a \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{a} & \frac{b}{a} \\ -\frac{\bar{b}}{a} & \frac{1}{\bar{a}} \end{pmatrix},$$

which is used in some problems of differential geometry [6] but neither S nor \tilde{S} are generated by this mapping.

Now we show why the universal gates can be constructed via a potential $Q(x)$ and particle's momentum k .

Scattering data of the equation (1) are defined as a set $\mathcal{S}^+ = \{R(k) = \frac{b(k)}{a(k)}, (k_j, b_j)_{j=1}^N\}$, where $R(k)$ is a meromorphic function in the upper half plane, $k_j = i\eta_j, j = 1, \dots, N$ is the finite set of poles of $R(k)$ and $b_j = \frac{\varphi(x, i\eta_j)}{\psi(x, i\eta_j)}$. From these scattering data \mathcal{S}^+ we can recover the scattering matrix, i. e. its unknown element $T(k) = \frac{1}{a(k)}$ in the following way

$$-\ln T(k) = \ln a(k) = \sum_{j=1}^N \ln \frac{k - i\eta_j}{k + i\eta_j} - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\ln(1 - |R(\zeta)|^2)}{\zeta - k - i0} d\zeta$$

or

$$T(k) = \prod_{j=1}^N \frac{k + i\eta_j}{k - i\eta_j} e^{\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\ln(1 - |R(\zeta)|^2)}{\zeta - k - i0} d\zeta}$$

Since by the Plemelj formula $\frac{1}{\zeta - k - i0} = \text{v.p.} \frac{1}{\zeta - k} + i\pi\delta(\zeta - k)$ ([3]. 50p), then

$$-\ln T(k) = -\frac{1}{2} \ln(1 - |R(k)|^2) + \sum_{j=1}^N \ln \frac{k - i\eta_j}{k + i\eta_j} - \frac{1}{2\pi i} \text{v.p.} \int_{\mathbb{R}} \frac{\ln(1 - |R(\zeta)|^2)}{\zeta - k} d\zeta$$

and finally we have

$$T(k) = \sqrt{1 - |R(k)|^2} \prod_{j=1}^N \frac{k + i\eta_j}{k - i\eta_j} \exp\left(\frac{1}{2\pi i} \text{v.p.} \int_{\mathbb{R}} \frac{\ln(1 - |R(\zeta)|^2)}{\zeta - k} d\zeta\right)$$

Evidently $|T(k)| = \sqrt{1 - |R(k)|^2}$, since $|k + i\eta| = |k - i\eta|$ and $\frac{1}{2\pi i} \text{v.p.} \int_{\mathbb{R}} \frac{\ln(1 - |R(\zeta)|^2)}{\zeta - k} d\zeta$ belongs to $[0, 1]$. Thus we obtain that the phase of $T(k)$ is defined by the Hilbert transform of the function $\ln(1 - |R(\zeta)|^2)$. Suppose now the sets of numbers $\{k_1, \dots, k_N\}$, $\{\alpha_1, \dots, \alpha_N\}$ and $\{\beta_1, \dots, \beta_N\}$ are given and for simplicity assume $k_j > 0$. One can find a continuous function $f(k), k > 0$ such that $f(k_j) = \alpha_j$ and $\gamma_j \doteq \frac{1}{\pi} \int_0^\infty \frac{f(\zeta)}{k_j - \zeta} d\zeta > \beta_j$. Also we can choose a continuous $g(k), k \leq 0$ such that $\frac{1}{\pi} \int_{-\infty}^0 \frac{g(\zeta)}{k_j - \zeta} d\zeta = \beta_j - \gamma_j$.

Thus the function

$$F(k) = \begin{cases} f(k), k > 0 \\ g(k), k \leq 0 \end{cases}$$

acquires the property

$$F(k_j) = \alpha_j, \quad \frac{1}{\pi} \int_{\mathbb{R}} \frac{F(\zeta)}{k_j - \zeta} d\zeta = \beta_j.$$

Using those relations for a given sequence $k_j, t_j, r_j, j = 1, \dots, N$, such that $|t_j|^2 = |r_j|^2$ we can construct a pair of functions $(T(k), R(k))$ such that $T(k_j) = t_j, R(k_j) = r_j$, since for $\alpha_j = t_j, \beta_j = \ln(1 - |r_j|^2)$ we find $F(k)$ and $T(k) = \frac{1}{\pi} \text{v.p.} \int_{\mathbb{R}} \frac{F(\zeta)}{k_j - \zeta} d\zeta = \beta_j$ and further define $R(k)$ by the equation $|R(k)|^2 = 1 - e^{-F(k)}$.

The potential $Q(x)$ can be recovered in the following way (see [2]). If

$$C(z) = -i \sum_{j=1}^N \gamma_j e^{-\eta_j} + \frac{1}{2\pi} \int_{\mathbb{R}} R(\xi) e^{i\xi x} d\xi,$$

where $\gamma_j = \frac{b_j}{a'(i\eta_j)}$, then Q is defined by the solution of the Gelfand-Levitan integral equation

$$K(x, y) + C(x + y) + \int_x^\infty K(x, s) C(s + y) ds = 0, \quad y > x$$

as $Q(x) = 2 \frac{d}{dx} K(x, x)$.

Therefore we have proved

Proposition 1. Let k_1, k_2, \dots, k_m be a sequence of pairwise distinct numbers and S_1, S_2, \dots, S_m be matrices from the set $\tau(SU(1, 1))$. Then there exists a potential $Q(x)$ such that the corresponding scattering matrix satisfies the conditions $S(k_1) = S_1, S(k_2) = S_2, \dots, S(k_m) = S_m$.

Now consider a particle with spin moving on the line in presence of an electromagnetic field $(E(x), B(x)), x \in \mathbb{R}$, where the electric field $E(x)$ is directed along the line and magnetic field $B(x)$ is directed across the line.

Let $A(x)$ be the potential of the magnetic field $B(x)$ and $Q(x)$ be the potential of the electric fields, i. e. $A'(x) = B(x), Q'(x) = -E(x)$.

The Schrödinger operator (as Hamiltonian) of such system has the form

$$(3) \quad \mathcal{H} = -\left(\frac{d^2}{dx^2} - A^2(x) + Q(x)\right)\mathbb{I} - B(x)\sigma_3,$$

where $\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is third Pauli matrix.

Here for $\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \in L^2(\mathbb{R}) \otimes \mathbb{C}$ we have

$$\mathcal{H}\psi = - \begin{pmatrix} \psi_1''(x) + (Q(x) + A'(x) - A(x)^2)\psi_1(x) \\ \psi_2''(x) + (Q(x) - A'(x) - A(x)^2)\psi_2(x) \end{pmatrix}.$$

Remark 3. Maybe it is more plausible to consider the behavior particle in the plane. In this case the potential of magnetic field is of the form $a(x, y) = (a_1(x, y), a_2(x, y))$ and in the gauge $a_1 = 0$ we can take $a_2'(x) = B(x)$. Then Hamiltonian can be written as

$$\mathcal{H} = \left(-\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + ia_2(x)\frac{\partial}{\partial y} - Q(x) + a_2^2(x)\right)\mathbb{I} - B(x)\sigma_3.$$

If we take $A(x) = a_2(x)$ and consider the wave functions independent on y -variable then we get the Hamiltonian (3).

If in advance we take two arbitrary potentials $U(x)$ and $V(x)$ then $A(x)$ and $Q(x)$ defined by relation

$$A'(x) = \frac{1}{2}(U(x) - V(x)), \quad Q(x) = A^2(x) + \frac{1}{2}(U(x) + V(x))$$

satisfy the equations

$$Q(x) + A'(x) - A^2(x) = U(x), \quad Q(x) - A'(x) - A^2(x) = V(x).$$

Hence we have

$$-\mathcal{H}\psi(x) = \frac{d^2}{dx^2}\psi(x) + \begin{pmatrix} U(x) & 0 \\ 0 & V(x) \end{pmatrix} \psi(x) \equiv - \begin{pmatrix} H_U\psi_1(x) \\ H_V\psi_2(x) \end{pmatrix}.$$

By definition of the scattering operator it follows that the scattering operator for the Hamiltonian $\mathcal{H} = \begin{pmatrix} H_U & 0 \\ 0 & H_V \end{pmatrix}$ has the form $S = \begin{pmatrix} S_U & 0 \\ 0 & S_V \end{pmatrix}$, where S_U and S_V are scattering operators for H_U and H_V respectively.

Therefore on the k^2 -energy level the scattering matrix of the system (3), which is a 4×4 -matrix, can be represented as

$$(4) \quad S_H(k) = \begin{pmatrix} S_U(k) & 0 \\ 0 & S_V(k) \end{pmatrix},$$

where $S_U(k)$ and $S_V(k)$ are scattering matrices corresponding to the potentials of U and V respectively. Evidently they are 2×2 -matrices from $\tau(SU(1, 1))$.

Therefore it is proved

Proposition 2. Let $U(x)$ and $V(x)$ be two arbitrary potentials. Then there exists a electro-magnetic potential $(A(x), Q(x))$ such that

the Hamiltonian of (3) takes the form $\mathcal{H} = \begin{pmatrix} H_U & 0 \\ 0 & H_V \end{pmatrix}$, where $H_U = -\frac{d^2}{dx^2} - U(x)$ and $H_V = -\frac{d^2}{dx^2} - V(x)$. Moreover the scattering matrix for the stationary Schrödinger equation $\mathcal{H}\psi = k^2\psi$ can be represented as a matrix (4).

As a corollary we get that 4×4 -matrix from the example 2

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & e^{i\phi} \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

can be represented as the scattering matrix of the system governed by (3). Moreover we can take the potential $U(x) = 0$ in both cases and $V(x)$ as indicated in example 1 and 2.

3. TWO-LEVEL SYSTEM IN ELECTRIC FIELD

Let us consider behavior of a two-level system in an electric field. It can be described by the equation

$$(5) \quad i \frac{d}{dt} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \begin{pmatrix} \frac{\zeta}{E(t)} & E(t) \\ -\zeta & \end{pmatrix} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}$$

where A and B denote the amplitudes of first and second level respectively, $E(t)$ is the complex envelope of the electric field and ζ is the difference between support frequency and proper frequency. The free Hamiltonian here is $H_0 = \zeta \sigma_3$ and free dynamics is given by the unitary matrix $e^{-i\zeta t \sigma_3} = \begin{pmatrix} e^{-i\zeta t} & 0 \\ 0 & e^{i\zeta t} \end{pmatrix}$.

Let $\Psi(t)$ and $\Phi(t)$ be matrix solution of (5) with condition

$$(6) \quad \Psi(t) \sim \begin{pmatrix} e^{-i\zeta t} & 0 \\ 0 & e^{i\zeta t} \end{pmatrix}, t \rightarrow \infty \text{ and } \Phi(t) \sim \begin{pmatrix} e^{-i\zeta t} & 0 \\ 0 & e^{i\zeta t} \end{pmatrix}, t \rightarrow -\infty$$

respectively. They are unitary unimodular matrices, since $\begin{pmatrix} \frac{\zeta}{E(t)} & E(t) \\ \frac{\zeta}{E(t)} & -\zeta \end{pmatrix}$

is a Hermitian traceless matrix. Thus the so called monodromy matrix $M(\zeta) = \Psi^{-1}(t)\Phi(t)$ belongs to $SU(2)$ and can be represented as

$$M(\zeta) = \begin{pmatrix} a(\zeta) & -\bar{b}(\zeta) \\ b(\zeta) & \bar{a}(\zeta) \end{pmatrix} \text{ with } |a(\zeta)|^2 + |b(\zeta)|^2 = 1.$$

Let $U(t, \tau)$ be the fundamental matrix of (5), i. e.

$$U(\tau, \tau) = I, \quad i \frac{\partial}{\partial t} U(t, \tau) = \begin{pmatrix} \frac{\zeta}{E(t)} & E(t) \\ \frac{\zeta}{E(t)} & -\zeta \end{pmatrix} U(t, \tau), \quad t > \tau.$$

Evidently there exist matrices W_φ and W_ψ such that $\Psi(t) = U(t, 0)W_\Psi$, $\Phi(t) = U(t, 0)W_\Phi$. The relation (6) can be rewritten as

$$\lim_{t \rightarrow \infty} \Psi(t)^* e^{-\zeta t \sigma_3} = \mathbb{I} \quad \text{and} \quad \lim_{t \rightarrow -\infty} \Phi(t)^* e^{-\zeta t \sigma_3} = \mathbb{I}.$$

On the other hand by definition of wave operators we have [7]

$$W_\pm = \lim_{t \rightarrow \mp \infty} U(t, 0)^* e^{-\zeta t \sigma_3}.$$

Hence

$$\begin{aligned} W_+ &= \lim_{t \rightarrow -\infty} U(t, 0)^* e^{-\zeta t \sigma_3} = \lim_{t \rightarrow -\infty} W_\Phi(U(t, 0)W_\Phi)^* e^{-\zeta t \sigma_3} \\ &= \lim_{t \rightarrow -\infty} W_\Phi \Phi(t)^* e^{-\zeta t \sigma_3} = W_\Phi. \end{aligned}$$

Similarly $W_- = W_\Psi$. Thus we have

$$\begin{aligned} S(\zeta) &= W_-^{-1} W_+ = W_\Psi^{-1} W_\Phi \\ &= \Psi(t)^{-1} U(t, 0) U(t, 0)^{-1} \Phi(t) = \Psi(t)^{-1} \Phi(t) = M(\zeta) \end{aligned}$$

i.e.

$$S(\zeta) = M(\zeta) = \begin{pmatrix} a(\zeta) & -\bar{b}(\zeta) \\ b(\zeta) & \bar{a}(\zeta) \end{pmatrix}, \quad |a(\zeta)|^2 + |b(\zeta)|^2 = 1.$$

The function $a(\zeta)$ is the boundary value of a holomorphic function on the upper half plane with zeros ζ_j , $\text{Im} \zeta_j > 0$, $j = 1, \dots, N$. For $\zeta = \zeta_j$ there exists a solution $\chi(t)$ of (5) with condition $\chi(t) \sim \begin{pmatrix} 0 \\ e^{-k_j t} \end{pmatrix}$, $t \rightarrow -\infty$, $\chi(t) \sim \begin{pmatrix} e^{ik_j t} \\ 0 \end{pmatrix} d_j$, $t \rightarrow \infty$. The scattering data for the system (5) is the set $\{b(\zeta), \zeta_j, d_j, j = 1, \dots, N\}$. The function $a(\zeta)$ is defined as

$$a(\zeta) = e^{-\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\ln(1-|b(\zeta')|^2)}{\zeta - \zeta' - i0} d\zeta'} \prod_{l=1}^N \frac{\zeta - \zeta_l}{\zeta - \bar{\zeta}_l}.$$

The field $E(t)$ can be restored by scattering data as follows (see [4]); $E(t) = -2iK(t, t)$, where $K(t) = \begin{pmatrix} K_{11}(t) & K_{12}(t) \\ K_{21}(t) & K_{22}(t) \end{pmatrix}$ is a solution of the Gelfand-Levitan integral matrix equation

$$K(t, s) + \widehat{F}(t, s) + \int_t^\infty K(t, s) \widehat{F}(z + s) dz, \quad t < s$$

with

$$\widehat{F}(t) = \begin{pmatrix} 0 & F(t) \\ \bar{F}(t) & 0 \end{pmatrix}, \quad F(t) = \frac{1}{2\pi} \int_{\mathbb{R}} r(k) e^{ikt} dk + \sum_{j=1}^N m_j e^{ik_j t},$$

$$r(\zeta) = \frac{b(\zeta)}{a(\zeta)}, \quad m_j = \frac{d_j}{a'(\zeta_j)}.$$

As in previous case we define the function $b(\zeta)$ to obtain a desired scattering matrix $S(\zeta)$. Hence we have

Proposition 3. For the finite set of numbers $\{\zeta_1, \zeta_2, \dots, \zeta_m\}$ and matrices S_1, S_2, \dots, S_m from the set $SU(2)$ there exists a potential $Q(x)$ such that the corresponding scattering matrix satisfies the conditions $S(\zeta_1) = S_1, S(\zeta_2) = S_2, \dots, S(\zeta_m) = S_m$.

In particular cases we can construct the matrices of example 1 up to a constant: $iH = \frac{i}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and $e^{-i\varphi/2} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$.

Now we suppose that electric field acts on the pair of 2-level particles performing dipole-dipole interaction between dipolar particle states. We assume that the dipoles are oriented along electric fields and therefore the hamiltonian of the two-particle system is presented as $\mathcal{H}_{AB}(E(t), \Phi_{AB})$ [13], where

$$\mathcal{H}_{AB}(x, y) = (\mathcal{H}_A + x\hat{d}_A) \otimes I + I \otimes (\mathcal{H}_B + x\hat{d}_B) + y\hat{d}_A \otimes \hat{d}_B$$

and

$$\mathcal{H}_C = \begin{pmatrix} W_+^C & 0 \\ 0 & W_-^C \end{pmatrix} \quad \hat{d}_C = \begin{pmatrix} 0 & d_C \\ \bar{d}_C & 0 \end{pmatrix} \quad C = A, B$$

In the case of $\Phi_{AB} = y$ the Hamiltonian can be rewritten as 4×4 matrix

$$\mathcal{H}_{AB}(E(t), y) = \begin{pmatrix} W_+^A + W_+^B & d_A E(t) & d_A E(t) & y d_A d_B \\ \bar{d}_A E(t) & W_+^A + W_-^B & y d_A \bar{d}_B & d_B E(t) \\ \bar{d}_A E(t) & y \bar{d}_A d_B & W_-^A + W_+^B & d_B E(t) \\ y \bar{d}_A \bar{d}_B & \bar{d}_B E(t) & \bar{d}_B E(t) & W_-^A + W_-^B \end{pmatrix}.$$

Proposition 4. Let $d_A, d_B, W_+^A, W_+^B, W_-^A, W_-^B$ be arbitrary complex numbers. Then there exists a continuous electric pulse $E(t)$ and interacting potential $\Phi_{AB}(t)$ vanishing at infinity and such that the corresponding scattering matrix does not belong in $SU(2) \otimes SU(2)$ i. e. it is entangled operator.

Proof. It is easy to see that the hamiltonian $\hat{d}_A \otimes \hat{d}_B$ is not represented as a matrix $M \otimes I + I \otimes N$ for some $M, N \in su(2)$. Hence the matrix

$$F(T) = \exp\{iT\mathcal{H}_{AB}(0, 0)\} \exp\{-2iT\mathcal{H}_{AB}(x, y)\} \exp\{iT\mathcal{H}_{AB}(0, 0)\}$$

for some T and $y \neq 0$ can not be represented as an element of $SU(2) \otimes SU(2)$, as

$$F'(0) = i\mathcal{H}_{AB}(0, 0) - 2i\mathcal{H}_{AB}(x, y) + i\mathcal{H}_{AB}(0, 0)$$

also cannot be represented as a matrix $M \otimes I + I \otimes N$ for some $M, N \in su(2)$. It is easy to see that the S-matrix of the system

$$i \frac{d}{dt} Y(t) = \mathcal{H}_{AB}(E(t), \Phi_{AB}(t)) Y(t)$$

for the electric pulse $E(t) = x 1_{[-T, T]}(t)$ and $\Phi_{AB}(t) = y 1_{[-T, T]}(t)$ coincides with matrix of (7).

Since $SU(2) \otimes SU(2)$ is a closed subset of $U(4)$, for sufficiently small nonzero x the matrix (7) also is an entangled one, i. e. does not belong to $SU(2) \otimes SU(2)$, thus each continuous pulse $E_\varepsilon(t)$ $\Phi_{AB, \varepsilon}(t)$ close enough to $E(t), \Phi_{AB}(t)$ also generates an entangled operator. \square

This proposition and results of [8] allow us to tune the electric pulse to construct a system of universal gates.

4. MONODROMY OF FUCHSIAN SYSTEMS AND THE S-MATRIX

Let $\{s_1, \dots, s_n\}$ be complex numbers on \mathbb{C} and $s_i \neq s_j, i \neq j$. Consider the Fuchsian system

$$(8) \quad df = \omega f,$$

where ω is a meromorphic 1-form on \mathbb{C}

$$\omega = \left(\frac{A_1}{z - s_1} + \frac{A_2}{z - s_2} + \dots + \frac{A_n}{z - s_n} \right) dz,$$

and $A_j, j = 1, \dots, n$ are constant 2×2 -matrices. Let (f_1, f_2) be solutions of (8) in the neighborhood of $z_0 \in \mathbb{C} \setminus \{s_1, \dots, s_n\}$ and $\gamma_1, \dots, \gamma_n$ be the generators of $\pi_1(\mathbb{C} \setminus \{s_1, \dots, s_n\}, z_0)$. After the extension of (f_1, f_2) along the γ_j we obtain other solutions (f_1^j, f_2^j) of (8) and $(f_1, f_2) = M_j(f_1^j, f_2^j)$, where $M_j, j = 1, \dots, n$ are monodromy matrices. Conversely, for fixed data $(s_1, \dots, s_n, M_1, \dots, M_n)$ there exists a Fuchsian system (8) with given singular points $s_j, j = 1, \dots, n$ and monodromy matrices $M_j, j = 1, \dots, n$. We take monodromy matrices from $SU(2)$ and consider them as scattering matrices. The singular points s_1, \dots, s_n are considered as sources of energy and ω is the gauge potential induced from the given data $(s_1, \dots, s_n, M_1, \dots, M_n)$.

Now we intend to show how the S-matrix of the system may be expressed as a monodromy matrix of the corresponding Fuchsian system. First we recall that two differential equations $i \frac{d}{dt} \psi_1(t) = \mathcal{H}_1(t) \psi_1(t)$ and $i \frac{d}{dt} \psi_2(t) = \mathcal{H}_2(t) \psi_2(t)$ are called gauge equivalent if there exists a differentiable 1-parameter family of invertible matrices such that $\mathcal{H}_2(t) = i U'(t) U(t)^{-1} + U(t) \mathcal{H}_1(t) U(t)^{-1}$. It is easy to see that if $\psi_1(t)$ is a solution of the first equation then $\psi_2(t) = U(t) \psi_1(t)$ is the solution of the second equation. By this definition the Schrödinger equation of

type $i\frac{d}{dt}\psi_0(t) = (\mathcal{H}_0 + V(t))\psi_0(t)$ is gauge equivalent to the so-called interacting representation $i\frac{d}{dt}\psi_1(t) = V_1(t)\psi_1(t)$ via 1-parameter family $U(t) = e^{it\mathcal{H}_0}$, where $V_1(t) = e^{it\mathcal{H}_0}V(t)e^{-it\mathcal{H}_0}$.

Now we consider a field of type $E(t)e^{-i\omega t}$. Then the system (5) recasts

$$(9) \quad i\frac{d}{dt} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix} = \zeta\sigma_3 + \begin{pmatrix} 0 & E(t)e^{-i\omega t} \\ E(t)e^{i\omega t} & 0 \end{pmatrix} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}.$$

Since

$$\begin{pmatrix} e^{i\zeta t} & 0 \\ 0 & e^{-i\zeta t} \end{pmatrix} \begin{pmatrix} 0 & e^{-i\omega t} \\ e^{i\omega t} & 0 \end{pmatrix} \begin{pmatrix} e^{-i\zeta t} & 0 \\ 0 & e^{i\zeta t} \end{pmatrix} = \begin{pmatrix} 0 & e^{-i(\omega-2\zeta)t} \\ e^{i(\omega-2\zeta)t} & 0 \end{pmatrix},$$

the interacting representation of the system (9) takes the form

$$(10) \quad i\frac{d}{dt}\Phi(t) = \begin{pmatrix} 0 & E(t)e^{-i(\omega-2\zeta)t} \\ E(t)e^{i(\omega-2\zeta)t} & 0 \end{pmatrix} \Phi(t)$$

Let $\Phi(t, \tau)$ denote the fundamental matrix of (10). Then the S-matrix of the system (9) can be represented as

$$(11) \quad S(\zeta) = \lim_{\substack{t \rightarrow +\infty, \\ \tau \rightarrow -\infty}} \Phi(t, \tau).$$

In the particular case of analyticity of the function $E(t)$ the S-matrix coincides with the monodromy matrix of the corresponding Fushian system

Example 3. Consider the case $\omega = 2\zeta$, $E(t, a, b) = \frac{2ab}{t^2+a^2} \equiv \frac{ib}{t+ia} - \frac{ib}{t-ia}$, $a, b \in \mathbb{R}$. By the equality $\sigma_1 = H\sigma_3H^{-1}$ the system (10) is gauge equivalent to

$$i\frac{d}{dt}\Psi(t) = \frac{2ab}{t^2+a^2}\sigma_3\Psi(t).$$

The solution of the last equation is

$$\Psi(t) = \begin{pmatrix} \exp(-ib \int \frac{2adt}{t^2+a^2}) & 0 \\ 0 & \exp(ib \int \frac{2adt}{t^2+a^2}) \end{pmatrix}$$

and thanks to the formula (11) the S-matrix is $e^{-2\pi b \int \frac{2adt}{t^2+a^2} i\sigma_3} = e^{-bi\sigma_3}$. Since $\Phi(z) = H\Psi(z)$, one has

$$S(\zeta) = M(\gamma) = H \begin{pmatrix} e^{2\pi bit} & 0 \\ 0 & e^{-2\pi bit} \end{pmatrix} H^{-1} = e^{-2i\pi b\sigma_1}.$$

Now extend the range of the variable t to the complex plane \mathbb{C} and compute the monodromy matrix of the so obtained Fuchsian system.

The system (10) looks as follows

$$i \frac{d}{dw} \Phi(w) = \frac{2ab\sigma_1}{w^2 + a^2} \Phi(w),$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By the equation $\sigma_1 = H\sigma_3H^{-1}$ this system is gauge equivalent to

$$i \frac{d}{dw} \Psi(w) = \frac{2ab\sigma_3}{w^2 + a^2} \Psi(w).$$

The line $Imw = 0$ on the Riemann sphere $P_1(\mathbb{C}) \supset \mathbb{C}$ becomes a loop. In order to express this loop explicitly we make change of coordinate $z = \frac{1}{w-i}$, which is a holomorphic map $P_1(\mathbb{C}) \rightarrow P_1(\mathbb{C})$ with inverse $w = \frac{1}{z} + i$. This mapping carries the 1-form $\frac{2ab}{w^2+a^2}dw$ to the 1-form $-\frac{2ab}{a^2z^2+(iz+1)^2}dz$. In view of

$$-\frac{2ab}{a^2z^2+(iz+1)^2} = \frac{ib}{z - \frac{i}{a+1}} - \frac{ib}{z + \frac{i}{a-1}}$$

we get the Fushian system

$$(12) \quad \frac{d}{dz} \Psi(z) = \left(\frac{b\sigma_3}{z - \frac{i}{a+1}} - \frac{b\sigma_3}{z + \frac{i}{a-1}} \right) \Psi(z),$$

which has the solution

$$\Psi(z) = \begin{pmatrix} \exp\left(b \int \frac{dz}{z - \frac{i}{a+1}} - b \int \frac{dz}{z + \frac{i}{a-1}}\right) & 0 \\ 0 & \exp\left(-b \int \frac{dz}{z - \frac{i}{a+1}} + b \int \frac{dz}{z + \frac{i}{a-1}}\right) \end{pmatrix}.$$

In new coordinates the line $w = t$ becomes the curve $z = \frac{1}{t-i} \equiv \frac{t}{1+t^2} + i\frac{1}{1+t^2}$, which is circle $\gamma : x^2 + (y - \frac{1}{2})^2 = \frac{1}{4}$. The poles of system (12) $\frac{i}{a+1}$ and $\frac{i}{1-a}$ lie inside and outside of the circle respectively. Hence we can consider the circle γ as loop around pole $\frac{i}{a+1}$ and monodromy matrix of (12) coincides to S-matrix given by (11) Since for solution of (12) is $\Phi(z) = H\Psi(z) = H^{-1}\Psi(z)$, one has

$$S(\zeta) = M(\gamma) = H \begin{pmatrix} e^{2\pi bi} & 0 \\ 0 & e^{-2\pi bi} \end{pmatrix} H^{-1} = e^{-2\pi bi\sigma_1}.$$

□

If $E(t) = \sum \frac{2a_k b_k}{t^2 + a_k^2}$ then $S(\zeta) = \prod M(\gamma_k)$, where γ_k is a loop around $\frac{1}{1+a_k}$ and $M(\gamma_k)$ is the corresponding monodromy matrix.

Example 4. If we consider the field $E(t) = \frac{2t}{t^2+a^2}$, the integral $\int_{-\infty}^{\infty} \frac{2t dt}{t^2+a^2}$ exists only in the sense of principal value. Extending the field in the complex plane, after change of variable we obtain

$$(13) \quad \frac{d}{dz}\Phi(z) = \frac{1}{a} \left(\frac{1}{z} + i \right) \left(\frac{\sigma_1}{z - \frac{i}{a+1}} - \frac{\sigma_1}{z + \frac{i}{a-1}} \right) \Phi(z).$$

Hence the complementary pole $z = 0$ appears, which lies in γ . We pass to a gauge equivalent differential equation of type

$$\frac{d}{dz}\Psi(z) = \left(\frac{b_1}{z} + \frac{b_2}{z-p} + \frac{b_3}{z-q} \right) \sigma_3 \Psi(z), \quad p = \frac{i}{1+a}, \quad q = \frac{i}{1-a}$$

for some $b_1, b_2, b_3 \in \mathbb{C}$. The monodromy matrix of such system around γ is the matrix

$$\begin{pmatrix} e^{\int_{\gamma} (\frac{b_1}{z} + \frac{b_2}{z-p} + \frac{b_3}{z-q}) dz} & 0 \\ 0 & e^{-\int_{\gamma} (\frac{b_1}{z} + \frac{b_2}{z-p} + \frac{b_3}{z-q}) dz} \end{pmatrix}$$

which contains a Cauchy type integral v.p. $\int_{\gamma} \frac{b_1}{z} dz$. Since 0 lies on γ and p lies inside γ , this matrix is equal to

$$\begin{pmatrix} \exp(i\pi b_1 + 2\pi i b_2) & 0 \\ 0 & \exp(-i\pi b_1 - 2\pi i b_2) \end{pmatrix} = \exp(i\pi(b_1 + 2b_2)\sigma_3).$$

Hence the system (13) has the monodromy matrix $\exp(i\pi(b_1 + 2b_2)\sigma_1)$ similarly to the previous example.

Remark 4. In the case of $\omega \neq 2\zeta$ the more complicated irregular singularities emerge. We intend to study them in subsequent works.

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